Time–Entropy Duality and the Lax Double Category of Thermodynamics

II. Relativistic Extension, Horizons, and Black-Hole Entropy

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Abstract

In Part I [1] we introduced a lax double category $\mathbb{T}\mathrm{herm}$ of thermodynamic states, in which reversible (time-like) dynamics appear as horizontal morphisms, irreversible (dissipative) dynamics as vertical morphisms, and the failure of the two to commute is measured by a non-invertible comparison cell with cost $\mathcal{A}(\chi)$—the *categorical mixing cost*. The present paper extends this framework to relativistic systems.

We establish: (1) a special-relativistic extension as a *frame-indexed* family of double categories satisfying the cocycle condition—a *prestack*; the cocycle condition is proved for inertial frames from the Lorentz group law, while on 4-velocities the canonical (rotation-free) boosts violate the cocycle by a Wigner rotation, making the velocity-indexed structure gerbe-like; the stack property (effective descent) remains open; (2) an explicit construction for a Klein–Gordon field with BDNK-inspired (frame-tied) dissipation, exhibiting the observer-dependence of the comparison cell cost; (3) a connection to the Unruh effect, in which the Unruh temperature $T_U = a/(2\pi)$ appears as the comparison cell cost of the accelerated observer; (4) an extension to Schwarzschild spacetime in which the regularized horizon-integrated cost reproduces the Bekenstein–Hawking entropy, $\mathcal{A}_{\mathrm{HH}}(\chi) = A_{\mathcal{H}}/(4G) = S_{\mathrm{BH}}$, under a normalization consistent with Wald's Noether-charge formula; we prove that this normalization is at present a *choice* matched to Wald's formula, not an independent derivation, and formulate the independent derivation as a sharp open problem; and (5) an $(\infty, 2)$-categorical conjecture explaining logarithmic corrections to black-hole entropy as contributions from higher cells in the Baker–Campbell–Hausdorff hierarchy.

Several of the gaps admit partial closure, which this version carries out: the Wigner 2-cocycle is exhibited as a coboundary (the gerbe is trivial, but no Lorentz-equivariant trivialization exists, so strictification costs manifest covariance); the kinematic half of the non-inertial cocycle is proved by the torsor argument; the per-mode comparison cell of the damped field oscillator is computed exactly at the Gaussian level, deriving the T_U-proportionality and the horizon localization within a shell model; the modular-flow conjecture is proved in every finite-dimensional truncation; and a no-go theorem shows that the Bekenstein–Hawking normalization cannot be derived from matter-sector data alone, reformulating the normalization problem as an induced-gravity computation.

The paper is self-contained: Section 2 summarizes the framework of Part I in the form used here. All numerical results have been independently confirmed by

computer.

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1 Introduction

Part I [1] of this work developed the *Duality Proposition*—time and entropy are dual quantities derived from the symmetry structure of a single path-space action functional—and made it precise through a lax double category $\mathbb{T}\mathrm{herm}$ of thermodynamic states, verified in detail for finite-state Markov chains. Two quantities emerged: the categorical mixing cost $\mathcal{A}(\chi)$, measuring the non-commutativity of the reversible and dissipative flows, and the housekeeping entropy production σ_{hk}; on the generic locus they vanish at orders ε^1 and ε^2 respectively in the distance ε from equilibrium.

That analysis was non-relativistic: it presupposed a global time parameter and an observer-independent notion of dissipation. Both assumptions fail in relativity, and their failure is not a technicality. An observer-dependent entropy threatens to introduce a second observer-dependence over and above Lorentz covariance—a difficulty open since Connes–Rovelli [2]—while quantum field theory in curved spacetime assigns *different thermal contents to the same state for different observers* (Unruh effect [3]), and black-hole thermodynamics ties entropy to horizon geometry (Bekenstein [4], Hawking [5], Wald [6])—indeed the Einstein equations themselves admit a thermodynamic derivation (Jacobson [12]).

This paper extends $\mathbb{T}\mathrm{herm}$ to exactly these settings. The guiding principle is that the structure that was global in Part I becomes *indexed by observers*: each observer u carries its own double category $\mathbb{T}\mathrm{herm}_u^{\mathrm{rel}}$, and the physics lies in the compatibility functors between them. The central results are:

- the frame-indexed family is a *prestack* (cocycle condition proved for inertial frames; on 4-velocities the Wigner rotation obstructs the cocycle and the structure is gerbe-like; Section 4);
- for a uniformly accelerated observer the comparison cell cost is proportional to the Unruh temperature $T_U = a/(2\pi)$ (Section 6);
- for Schwarzschild spacetime, the horizon-integrated cost reproduces $S_{\mathrm{BH}} = A_{\mathcal{H}}/(4G)$ under a Wald-consistent normalization, whose current circularity we prove and isolate as the key open problem (Section 7);
- logarithmic corrections to S_{BH} are conjecturally higher-cell contributions in an $(\infty, 2)$-categorical refinement $\mathbb{T}\mathrm{herm}^{\infty}$ (Section 8).

Throughout we use natural units $\hbar = c = k_B = 1$.

Methodological remark. We distinguish carefully between three grades of result: *proved* statements (stated as theorems and lemmas with complete proofs), *structural* claims (equalities of integrands or of formal structures, proved at that level but resting on an unproved quantum-field-theoretic evaluation), and *conjectures*. The unproved steps are displayed in orange boxes and collected as open problems R1–R8 in Section 9.2.

2 The Framework of Part I in Brief

This section summarizes, in self-contained form, the definitions and results of Part I [1] that the relativistic extension builds on. Proofs are given in Part I; here we restate them precisely.

2.1 The lax double category $\mathbb{T}\mathrm{herm}$

Definition 2.1: The lax double category $\mathbb{T}\mathrm{herm}$

- **Objects**: thermodynamic states (M, ρ)—a state space M with reference (stationary) measure π and a probability distribution ρ.
- **Horizontal morphisms**: reversible maps preserving the χ^2-divergence, $D_{\chi^2}(h \cdot \rho \| \pi) = D_{\chi^2}(\rho \| \pi)$; for finite-state Markov chains these are the orthogonal flows $h_t = e^{-tH_A}$ in the GNS representation (below).
- **Vertical morphisms**: stochastic kernels contracting relative entropy, $D_{\mathrm{KL}}(v \cdot \rho \| \pi) \le D_{\mathrm{KL}}(\rho \| \pi)$; generated by $-H_S$.
- **Two-cells**: compatibility squares with cost $\mathcal{A}(\square) \in [0, \infty]$.

Horizontal and vertical compositions are strict; the interchange law holds only *laxly*: there is a non-invertible comparison cell $\chi_{t,s}\colon h_t v_s \Rightarrow v_s h_t$ (the lax, non-invertible counterpart of the invertible comparison cells of pseudo-double categories in the sense of Grandis–Paré [22]).

2.2 The GNS split and the mixing cost

For a finite-state Markov chain with generator L and stationary distribution $\pi > 0$, the GNS (Gelfand–Naimark–Segal) representation conjugates the dynamics into a single Euclidean space: with $\Pi = \mathrm{diag}(\pi)$ and $\psi = \Pi^{-1/2}\rho$,

$$H := -\Pi^{-1/2} L \, \Pi^{1/2} = H_S + H_A, \qquad H_S = \tfrac{1}{2}(H + H^T), \quad H_A = \tfrac{1}{2}(H - H^T). \tag{1}$$

Key facts (Part I, Section 6): $H_A\sqrt{\pi} = H_S\sqrt{\pi} = 0$; e^{-tH_A} is orthogonal, preserves total mass and D_{χ^2} exactly; e^{-sH_S} contracts D_{KL}. The Baker–Campbell–Hausdorff (BCH) expansion

$$h_t \, v_s = v_s \, h_t \cdot \exp\Big(\tfrac{ts}{2}[H_A, H_S] + O(t^2 s, t s^2) \Big) \tag{2}$$

identifies the comparison cell, with *categorical mixing cost*

$$\mathcal{A}(\chi) = \tfrac{1}{2} \, \| [H_A, H_S] \|_F \cdot t \, s + O(t^2 s, t s^2). \tag{3}$$

The cell satisfies exact all-order interchange cocycle identities, of which the lax pentagon (bilinearity) and lax hexagon (Jacobi) coherence conditions are the leading linearizations (Part I, Section 6).

2.3 Order separation

Two further quantities enter. The *housekeeping entropy production* of the stationary fluxes $J_{ij} = \pi_j L_{ij}$ is $\sigma_{\mathrm{hk}} = \frac{1}{2}\sum_{i \neq j}(J_{ij} - J_{ji}) \ln(J_{ij}/J_{ji}) \ge 0$, vanishing exactly at detailed balance; the *cycle affinity* ε is the logarithm of the ratio of forward to backward rate products around the fundamental cycle, and $\varepsilon = 0$ is equivalent to detailed balance (Kolmogorov's criterion). Let ε measure the distance from the detailed-balance manifold $\{H_A = 0\}$. On the generic locus $\{[H_A, H_S] \neq 0\}$ (Part I, Section 6):

$$\mathcal{A}(\chi) = c_1 \varepsilon + O(\varepsilon^2), \qquad \sigma_{\mathrm{hk}} = c_2 \varepsilon^2 + O(\varepsilon^3), \tag{4}$$

with universal exponents; the zero sets are nested, $\{H_A = 0\} \subsetneq \{[H_A, H_S] = 0\}$. $\mathcal{A}(\chi)$ is thus the first-order (linear-response) order parameter of non-equilibrium physics, and is the quantity we carry into the relativistic setting.

Quantum order separation (Part I, Section 8). The framework transfers verbatim to Lindblad dynamics through the KMS-superoperator involution $\Theta(\mathcal{L}) = -\mathcal{L}^{\dagger_\sigma}$, and there the separation is *modified* in a way that matters below: quantum detailed balance does *not* kill the antisymmetric part—it leaves the coherent rotation $\mathcal{H} = -i[H, \cdot]$ in the reversible direction—so the mixing cost acquires a non-zero *equilibrium* value $\mathcal{A}(\chi_Q)(0) = \frac{1}{2}\|[\mathcal{H}, \mathcal{D}]\|$, and coherence parity suppresses the first-order response. The exponents shift from $(1, 2)$ to $(0, 2)$ in the coherent case (the secular case recovers $(1, 2)$). *A quantum system pays a categorical mixing cost even at detailed balance.* This zeroth-order equilibrium cost is the structural template for everything that follows: the Unruh cost of the accelerated observer (Section 6) and the Hartle–Hawking horizon cost (Section 7) are both equilibrium costs of KMS states, not driving responses.

2.4 The dictionary for what follows

In each relativistic extension the same triad recurs, with the following dictionary:

Part I (Markov chains)	Rindler (§6)	Schwarzschild (§7)
H_A (rotation)	boost generator K	modular Hamiltonian of the horizon wedge
H_S (dissipation)	BDNK dissipative Liouville	dissipative Liouville in HH vacuum
$[H_A, H_S]$	$[\mathcal{L}^{\mathrm{Ham}}, \mathcal{L}^{\mathrm{diss}}]$	$[\mathcal{L}^{\mathrm{Ham}}, \mathcal{L}^{\mathrm{diss}}]$, localized on $\mathcal{H}$
$\mathcal{A}(\chi) \propto ts$	$\propto T_U = a/(2\pi)$	$\propto A_{\mathcal{H}}/(4G)$
equilibrium cost $\mathcal{A}(\chi_Q)(0) = \frac{1}{2}\|[\mathcal{H}, \mathcal{D}]\|$	$\propto T_U$ in $\|0_M\rangle$	$= A_{\mathcal{H}}/(4G)$ in $\|0_{HH}\rangle$

3 Difficulties of the Relativistic Extension

Five structural obstacles separate the non-relativistic framework from a relativistic one:

(C1) *Loss of absolute time.* There is no global t to parametrize horizontal morphisms; each observer carries its own proper time.

(C2) *Observer-dependent entropy.* The vacuum of one observer is a thermal state for another (Unruh effect); "the" entropy of a state is not observer-invariant.

(C3) *Causal structure.* Morphisms must respect the light-cone structure; stochastic kernels acting instantaneously on spacelike slices are frame-dependent objects.

(C4) *Relativistic dissipation.* Naive relativistic Navier–Stokes is acausal and unstable. The BDNK (Bemfica–Disconzi–Noronha–Kovtun) first-order viscous framework [7, 8] shows that causal, stable first-order dissipation exists *provided* the dissipative terms are tied to a choice of frame u^μ; we adopt this structural lesson, implementing it for a scalar field by a frame-tied friction term (Section 5, Remark 5.1).

(C5) *Energy–momentum mixing.* Boosts mix energy density and momentum flux, so the split of a generator into "Hamiltonian" and "dissipative" parts is itself observer-dependent.

(C4) is resolved by adopting BDNK. (C1), (C2), and (C5) jointly dictate the shape of the construction: nothing observer-independent can be a single double category, so the extension must be an *observer-indexed family* with compatibility functors. (C3) constrains the admissible vertical morphisms within each fiber.

4 Special-Relativistic Extension: the Observer-Indexed Prestack

4.1 Observer-indexed double categories

Let $\mathcal{G} = SO^+(3,1)$ (proper orthochronous Lorentz group) and let $\mathcal{O}$ denote the space of *observers*, by which we mean inertial *frames*: orthonormal tetrads $u = (e_0, e_1, e_2, e_3)$ with e_0 future-directed timelike. The group $\mathcal{G}$ acts freely and transitively on tetrads, so $\mathcal{O}$ is a $\mathcal{G}$-torsor. The space of 4-*velocities* alone is the quotient $\mathcal{O}/SO(3)$—a homogeneous space whose stabilizer is the spatial rotation group, *not* a torsor; the distinction is essential (Remark 4.1 below). To each $u \in \mathcal{O}$ assign a lax double category $\mathbb{T}\mathrm{herm}_u^{\mathrm{rel}}$: its objects are states of the matter system as described in the frame u, horizontal morphisms are generated by the Hamiltonian Liouville operator $\mathcal{L}_u^{\mathrm{Ham}}$ (evolution in u-proper time), and vertical morphisms by the dissipative Liouville operator $\mathcal{L}_u^{\mathrm{diss}}$ (BDNK-type dissipation relative to u).

Definition 4.1: Special-relativistic structure $\mathbb{T}\mathrm{herm}^{\mathrm{SR}}$

$\mathbb{T}\mathrm{herm}^{\mathrm{SR}} := \{\mathbb{T}\mathrm{herm}_u^{\mathrm{rel}}\}_{u\in\mathcal{O}}$ together with inter-observer functors $F_{(u,u')}\colon \mathbb{T}\mathrm{herm}_u^{\mathrm{rel}} \to \mathbb{T}\mathrm{herm}_{u'}^{\mathrm{rel}}$ satisfying the *cocycle condition*

$$F_{(u',u'')} \circ F_{(u,u')} = F_{(u,u'')}, \qquad F_{(u,u)} = \mathrm{id}. \tag{5}$$

A family satisfying (5) is a *prestack* (of double categories) over $\mathcal{O}$. It is a *stack* if, in addition, every descent datum over a cover of $\mathcal{O}$ glues to a unique global object (effective descent).

4.2 The cocycle condition: proofs

The cocycle condition is not an axiom to be postulated but a property to be verified for the physically given $F_{(u,u')}$. For inertial observers it follows from the group structure of Lorentz transformations:

Lemma 4.1 (Trivial cocycle)**.** *If $F_{(u,u')} = \mathrm{id}$ for all u, u', the cocycle condition (5) holds.*

Proof. $F_{(u',u'')} \circ F_{(u,u')}(\rho) = \mathrm{id}(\mathrm{id}(\rho)) = \rho = F_{(u,u'')}(\rho)$, and $F_{(u,u)} = \mathrm{id}$ by hypothesis. □

Theorem 4.1: Group actions on torsors give cocycles

Let a group $\mathcal{G}$ act on a double category $\mathbb{C}$ by automorphisms, $\mathrm{act}\colon \mathcal{G} \to \mathrm{Aut}(\mathbb{C})$, and let $\mathcal{O}$ be a right $\mathcal{G}$-torsor: for $u, u' \in \mathcal{O}$ there is a *unique* $g_{u,u'} \in \mathcal{G}$ with $u' = u \cdot g_{u,u'}$. Define $F_{(u,u')} := \mathrm{act}\big(g_{u,u'}^{-1}\big)$ (the passive transformation re-expressing data in the new frame). Then the cocycle condition (5) holds.

Proof. The transition elements satisfy $g_{u,u} = e$ and the composition identity $g_{u,u'}\, g_{u',u''} = g_{u,u''}$, by uniqueness: $u \cdot (g_{u,u'} g_{u',u''}) = (u \cdot g_{u,u'}) \cdot g_{u',u''} = u' \cdot g_{u',u''} = u''$. Since act is a homomorphism,

$$\begin{aligned} F_{(u',u'')} \circ F_{(u,u')} &= \mathrm{act}\big(g_{u',u''}^{-1}\big)\,\mathrm{act}\big(g_{u,u'}^{-1}\big) = \mathrm{act}\big(g_{u',u''}^{-1}\, g_{u,u'}^{-1}\big) \\ &= \mathrm{act}\big((g_{u,u'}\, g_{u',u''})^{-1}\big) = \mathrm{act}\big(g_{u,u''}^{-1}\big) = F_{(u,u'')}, \end{aligned}$$

and $F_{(u,u)} = \mathrm{act}(e) = \mathrm{id}$. □

Theorem 4.2: $\mathbb{T}\mathrm{herm}^{\mathrm{SR}}$ is a prestack (frame-indexed)

$\mathbb{T}\mathrm{herm}^{\mathrm{SR}}$ of Definition 4.1, indexed by inertial *frames* and with inter-observer functors $F_{(u,u')} = \mathrm{act}(g_{u,u'}^{-1})$ implemented by the (unitary representation of the) unique Lorentz transformation relating the tetrads, is a prestack. The stack property (effective descent) is not established.

Proof. The space of tetrads is a $\mathcal{G}$-torsor, so the cocycle condition is exactly the content of Theorem 4.1. Effective descent would additionally require a Grothendieck topology on $\mathcal{O}$ and a gluing theorem for the quantum-field data $(\mathcal{L}_u^{\mathrm{Ham}}, \mathcal{L}_u^{\mathrm{diss}})$; neither is constructed here. The claim is therefore precisely "prestack," no more. □

Remark 4.1 (Wigner rotation: velocity indexing is gerbe-like)**.** *The frame indexing is not a pedantic choice. On the quotient $\mathcal{O}/SO(3)$ of 4-velocities, the natural section assigns to each pair (v, v') the canonical (rotation-free) pure boost $b_{v,v'}$—but pure boosts do not compose:*

$$b_{v',v''}\, b_{v,v'} \;=\; b_{v,v''} \cdot W(v, v', v''), \tag{6}$$

with $W(v, v', v'') \in SO(3)$ the Wigner rotation*, non-trivial whenever the three velocities are not coplanar with the time axis. Hence the velocity-indexed family with canonical boosts satisfies the cocycle condition only up to a rotation-valued 2-cocycle: it is* not *a prestack, but a gerbe-like structure banded by $SO(3)$. Passing to the frame bundle trivializes this 2-cocycle, which is what Theorem 4.2 exploits.*

The gerbe question raised by Remark 4.1 can in fact be settled completely:

Theorem 4.3: Strictification of the Wigner gerbe

Let $H^3 = \mathcal{O}/SO(3)$ be the velocity hyperboloid and fix a reference velocity v_0.

(i) **The gerbe is trivial.** The principal $SO(3)$-bundle $SO^+(3,1) \to H^3$ admits a global smooth section—explicitly, the rotation-free boosts $v \mapsto B_v := b_{v_0,v}$ from the reference velocity—since H^3 is contractible. Consequently the Wigner 2-cocycle is a coboundary: with $\widetilde{W}(v, v') := B_{v'}^{-1}\, b_{v,v'}\, B_v \in SO(3)$,

$$b_{v,v''}^{-1}\, b_{v',v''}\, b_{v,v'} \;=\; B_v\, \widetilde{W}(v, v'')^{-1}\, \widetilde{W}(v', v'')\, \widetilde{W}(v, v')\, B_v^{-1}. \tag{7}$$

(ii) **Explicit strictification.** The transitions $g_{v,v'} := B_v^{-1} B_{v'}$ satisfy the strict cocycle condition $g_{v,v'} g_{v',v''} = g_{v,v''}$ identically, so the velocity-indexed family *can* be made a prestack—at the price of routing all transitions through the fixed reference frame.

(iii) **No equivariant trivialization.** There is no Lorentz-equivariant section: a section $s\colon H^3 \to SO^+(3,1)$ with $s(\Lambda v) = \Lambda s(v)$ for all Λ would satisfy, at $v = v_0$ and $\Lambda = R \in SO(3)$ stabilizing v_0, $s(v_0) = R\, s(v_0)$ for every R, which is impossible.

Proof. (i) $H^3 \cong \mathbb{R}^3$ is contractible, so every principal bundle over it is trivial; the rotation-free boosts provide the section explicitly, and $\widetilde{W}(v, v')$ fixes v_0 (it maps $v_0 \to v \to v' \to v_0$), hence lies in $SO(3)$. Equation (7) follows by substituting $b_{v,v'} = B_{v'} \widetilde{W}(v, v') B_v^{-1}$ into the left-hand side and cancelling. (ii) $g_{v,v'} g_{v',v''} = B_v^{-1} B_{v'} B_{v'}^{-1} B_{v''} = B_v^{-1} B_{v''}$. (iii) As stated. All identities are confirmed to machine precision. □

Remark 4.2 (Covariance trade-off)**.** *Theorem 4.3 says the obstruction encoded in the Wigner rotation is not topological but* equivariant*: the velocity-indexed structure can be strictified, but only by choosing a fiducial frame and thereby breaking manifest Lorentz covariance; the covariant alternative is frame indexing (Theorem 4.2). This is the categorical form of a familiar physical dichotomy—fix a laboratory frame, or carry the frame as part of the data.*

4.3 Accelerated observers: kinematic vs. dynamical content of R2

Frame indexing also splits the non-inertial problem into a solved and an unsolved half:

Lemma 4.2 (Kinematic cocycle along accelerated worldlines)**.** *Let $u(\tau)$ be the family of instantaneous rest frames (tetrads) along any smooth, accelerated worldline. The kinematic transition functors $F^{\mathrm{kin}}_{(u(\tau),u(\tau'))} = \mathrm{act}\big(g^{-1}_{u(\tau),u(\tau')}\big)$ satisfy the cocycle condition exactly.*

Proof. The instantaneous tetrads are points of the same torsor $\mathcal{O}$; Theorem 4.1 applies verbatim to the restriction of the torsor structure to the curve. □

What does *not* follow is the cocycle for the full, *dressed* functors $F = F^{\mathrm{kin}} \circ \mathcal{B}$, where $\mathcal{B}$ implements the change of Fock representation (Bogoliubov dressing) between the observers' vacua. At the structural level the same torsor argument applies once more: Bogoliubov maps form a group (the restricted symplectic group), and dressed data indexed by *framed polarizations* again forms a torsor-like object, so the cocycle extends to any sector in which the dressings are unitarily implemented. The true content of R2 is therefore the *implementability* (Shale condition) of the wedge-to-wedge Bogoliubov maps—which is precisely where the type-III character of the wedge algebra enters.

Open gap (proof sketch only)

Non-inertial cocycle: remaining core (open problem R2). The kinematic half is settled (Lemma 4.2); what remains is the *dynamical dressing*: the unitary implementability (Shale condition) of the wedge-to-wedge Bogoliubov maps relating the accelerated observers' Fock representations, incorporating the Unruh–DeWitt re-

sponse $\mathcal{R}(\omega) = \omega/\big(2\pi(e^{2\pi\omega/a} - 1)\big)$. Between inequivalent representations (type III wedge algebras) implementability fails in the naive sense, and the correct formulation is part of the problem.

4.4 General-relativistic extension

Over a globally hyperbolic spacetime $(\mathcal{M}, g)$, observers form the bundle of oriented orthonormal frames (fiberwise a $\mathcal{G}$-torsor, the analogue of the torsor structure of §4.1), and $\mathbb{T}\mathrm{herm}^{\mathrm{GR}}$ is the corresponding bundle of double categories with parallel-transport functors along observer worldlines. The fiberwise statement of Theorem 4.2 applies verbatim; globally, two new phenomena appear. First, parallel transport around closed worldline families has non-trivial holonomy, which plays for $\mathbb{T}\mathrm{herm}^{\mathrm{GR}}$ the role that the Wigner rotation plays in Remark 4.1: velocity-indexed (rather than frame-indexed) data over curved spacetime is gerbe-like for the same reason, now sourced by curvature as well as by boost non-commutativity. Second, at a Killing horizon the norm of the Killing vector degenerates, $\xi \cdot \xi \to 0$, the local Tolman temperature diverges, and the comparison cell cost diverges pointwise; its *regularized horizon integral* is finite and is computed in Section 7.

5 Explicit SR Construction: Klein–Gordon Field with BDNK-Inspired Dissipation

5.1 Model

Take a real scalar field with Lagrangian $\mathcal{L}_{\mathrm{KG}} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$, coupled to frame-tied dissipation through an observer-dependent friction and noise:

$$(-\Box + m^2)\,\phi(x) = -\eta(x)\,u^\mu\partial_\mu\phi(x) + \xi(x), \tag{8}$$

where u^μ is the observer 4-velocity, η the friction coefficient, and ξ thermal noise satisfying the fluctuation–dissipation relation appropriate to the observer's KMS state.

Remark 5.1 (On the name "BDNK-inspired")**.** *Equation (8) is a covariant Langevin equation,* not *BDNK hydrodynamics: BDNK proper concerns first-order viscous corrections to the energy–momentum tensor of a fluid [7, 8]. What we import from BDNK is its structural lesson—causal, stable first-order dissipation exists, but only when the dissipative terms are tied to a choice of frame u^μ. Equation (8) is the minimal scalar-field implementation of that lesson: the friction $\eta\,u^\mu\partial_\mu\phi$ is first-order in derivatives, frame-tied, and damps the field causally for $\eta > 0$ and timelike u^μ. We therefore speak of BDNK-*inspired *dissipation.*

5.2 Double-category elements

For each observer u:

- **Horizontal**: the Hamiltonian Liouville operator $\mathcal{L}_u^{\mathrm{Ham}}$ generating evolution in u-proper time of the free field—the relativistic analogue of $-H_A$;

- **Vertical**: the dissipative Liouville operator $\mathcal{L}_u^{\mathrm{diss}}$ generated by the friction–noise pair of (8)—the analogue of $-H_S$.

Both depend explicitly on u^μ, realizing difficulty (C5): the very split into reversible and dissipative parts is observer-indexed.

5.3 Comparison cell cost

The BCH formula (2) applied to $(\mathcal{L}_u^{\mathrm{Ham}}, \mathcal{L}_u^{\mathrm{diss}})$ gives, at leading order in evolution times τ (horizontal) and s (vertical),

$$\mathcal{A}_u(\chi) \approx \frac{\tau s}{2}\, \eta \left\langle \mathcal{O}_u[\phi, \pi_\phi] \right\rangle_\rho, \tag{9}$$

where $\mathcal{O}_u$ is a quadratic form in the field and its conjugate momentum π_ϕ (not to be confused with the stationary measure π of §2) determined by $[\mathcal{L}_u^{\mathrm{Ham}}, \mathcal{L}_u^{\mathrm{diss}}]$. Since $\mathcal{O}_u$ depends on u, $\mathcal{A}_u(\chi) \neq \mathcal{A}_{u'}(\chi)$ in general: *the categorical mixing cost is observer-dependent*, exactly as the Unruh effect requires.

5.4 Lorentz consistency

For inertial observers the construction is consistent: the functors $F_{(u,u')}$ implementing Lorentz boosts satisfy the cocycle condition by Theorem 4.1, and they intertwine the pairs $(\mathcal{L}_u^{\mathrm{Ham}}, \mathcal{L}_u^{\mathrm{diss}})$ covariantly. The non-inertial sector is the open gap R2 of Section 4.

6 Connection to the Unruh Effect

6.1 Rindler coordinates and Killing structure

A uniformly accelerated observer with proper acceleration a follows $t(\tau) = a^{-1}\sinh(a\tau)$, $x(\tau) = a^{-1}\cosh(a\tau)$. In the right Rindler wedge, with coordinates (η, ξ), the metric is $ds^2 = -\xi^2 d\eta^2 + d\xi^2$ and the boost Killing vector is ∂_η; Rindler time translation *is* the Lorentz boost.

6.2 Boost generator as horizontal generator

The generator of Rindler-time translations for the Klein–Gordon field is the boost generator restricted to the wedge,

$$K = \int_0^\infty d\xi\, \xi \left[\tfrac{1}{2}\pi_R^2 + \tfrac{1}{2}(\partial_\xi \phi)^2 + \tfrac{1}{2} m^2 \phi^2\right], \tag{10}$$

conjugate to the *dimensionless* Rindler time η. In the dictionary of §2, K plays the role of H_A: it generates the reversible, horizontal direction of the accelerated observer's double category.

6.3 Unruh temperature as thermal structure

The Minkowski vacuum restricted to the wedge is thermal with respect to K [3, 9, 10]:

$$\rho_R^{(M)} = \frac{1}{Z}\, e^{-2\pi K}, \tag{11}$$

i.e. KMS at the dimensionless temperature $1/2\pi$ in boost time η. The observer following the worldline $\xi = 1/a$ has proper time $\tau = \eta/a$ and proper-time Hamiltonian $H_\tau = aK$, hence experiences the same state as thermal at

$$\boxed{T_U = \frac{a}{2\pi}}. \tag{12}$$

(The distinction matters dimensionally: K and η are dimensionless conjugates, H_τ and τ carry units; the Unruh temperature refers to the latter pair.) This is the KMS structure that, by the Duality Proposition of Part I, supplies the *entropic* symmetry direction for the accelerated observer.

6.4 Comparison cell cost of the accelerated observer

Apply (9) to the Rindler observer, with $\mathcal{L}^{\text{diss}}$ the BDNK dissipative Liouville constructed from the Unruh–DeWitt response at temperature T_U. The structural result is

$$\mathcal{A}_{\text{Rindler}}(\chi) \;\propto\; T_U \cdot f(a, m, \ldots) = \frac{a}{2\pi}\, f(a, m, \ldots), \tag{13}$$

where the proportionality to T_U is fixed by the KMS condition, while the dimensionless mode-sum factor f requires the full quantum-field computation (open problem R3). The next subsection derives the T_U-proportionality *exactly* at the Gaussian mode level, reducing R3 to the mode sum itself.

6.5 Exact Gaussian mode-level computation

Each Rindler mode of frequency ω, subject to the frame-tied friction of (8) and its FDT noise, is a damped harmonic oscillator. Its Gaussian dynamics closes on the covariance matrix $\sigma = \left(\begin{smallmatrix} \sigma_{xx} & \sigma_{xp} \\ \sigma_{xp} & \sigma_{pp} \end{smallmatrix}\right)$, which evolves by the affine flows

$$\dot\sigma = A\sigma + \sigma A^T + D, \qquad A_H = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \;\; A_D = \begin{pmatrix} 0 & 0 \\ 0 & -2\gamma \end{pmatrix}, \;\; D = \begin{pmatrix} 0 & 0 \\ 0 & 4\gamma\nu \end{pmatrix}, \tag{14}$$

where γ is the friction and ν the noise strength, fixed by the fluctuation–dissipation relation: $\nu = (\omega/2)\coth(\omega/2T)$ for a bath at temperature T, with stationary width $\sigma_{pp}^{\infty} = \nu$. The horizontal and vertical flows are the one-parameter groups generated by $(A_H, 0)$ and (A_D, D) in the affine group of covariance space, and the comparison cell is their group commutator, exactly as in Part I.

Lemma 6.1 (Per-mode comparison cell, exact)**.** *The Lie bracket of the two affine generators has linear part* $C = [A_H, A_D] = \left(\begin{smallmatrix} 0 & -2\gamma \\ -2\gamma\omega^2 & 0 \end{smallmatrix}\right)$ *(friction–rotation mixing, temperature-independent) and offset (thermal) part*

$$A_H D + D A_H^T = \begin{pmatrix} 0 & 4\gamma\nu \\ 4\gamma\nu & 0 \end{pmatrix}, \qquad \left\| A_H D + D A_H^T \right\|_F = 4\sqrt{2}\,\gamma\,\nu, \tag{15}$$

exactly linear *in the noise strength* ν. *The per-mode cost* $\mathcal{A}_\omega(\chi) = \frac{ts}{2}\,\|(C\text{-part},\ \text{offset})\| + O(t^2 s, ts^2)$ *therefore decomposes into a temperature-independent friction component and a thermal component* $\frac{ts}{2}\cdot 4\sqrt{2}\,\gamma\nu$.

Proof. Direct computation: $A_H A_D = \left(\begin{smallmatrix} 0 & -2\gamma \\ 0 & 0 \end{smallmatrix}\right)$, $A_D A_H = \left(\begin{smallmatrix} 0 & 0 \\ 2\gamma\omega^2 & 0 \end{smallmatrix}\right)$, and $A_H D = \left(\begin{smallmatrix} 0 & 4\gamma\nu \\ 0 & 0 \end{smallmatrix}\right)$. The Lie bracket in the affine algebra $\mathfrak{gl} \ltimes V$ of pairs (linear action on σ, offset) is $\big([A_H, A_D]\text{-action},\ A_H D + D A_H^T\big)$; the group commutator of the two flows reproduces it at order ts, confirmed numerically with $O(t)$ convergence of the residuals. □

Theorem 6.1: T_U-proportionality at the Gaussian level

For a Rindler mode at local temperature T, in the thermal regime $T \gg \omega$ (equivalently near the horizon, where the Tolman temperature diverges), the noise strength is $\nu = (\omega/2)\coth(\omega/2T) \to T$, and the thermal component of the per-mode comparison cell is

$$\mathcal{A}_\omega^{\text{th}}(\chi) = \frac{ts}{2}\cdot 4\sqrt{2}\,\gamma(\omega)\,T + O(\omega^2/T), \tag{16}$$

exactly linear in the temperature. Summed over Rindler modes at $T = T_U$, the thermal cost is $\mathcal{A}^{\text{th}}_{\text{Rindler}} \propto T_U \cdot \sum_\omega \gamma(\omega)$, with $\gamma(\omega)$ determined by the Unruh–DeWitt response: the structural claim (13) is thereby *derived* at the Gaussian level, and the mode-sum factor f is identified as $f \propto \sum_\omega \gamma(\omega)$—ultraviolet divergent, hence requiring exactly the cutoff ℓ_c of Section 7.

Proof. Immediate from Lemma 6.1 and $\coth x = 1/x + O(x)$: $\nu = T + O(\omega^2/T)$. Linearity of (15) in ν is exact (verified over two decades of ν in the test suite). □

This settles the structure of R3: what remains open is only the evaluation of the regularized mode sum $\sum_\omega \gamma(\omega)$ in the genuine (non-Gaussian, interacting) field theory.

Unruh effect as double-category structure. For the inertial observer in the Minkowski vacuum the dissipative direction is trivial and $\mathcal{A}_{\text{inertial}}(\chi)\big|_{|0_M\rangle} = 0$; for the Rindler observer *in the same state*, $\mathcal{A}_{\text{Rindler}}(\chi) \propto T_U \neq 0$. The observer-dependence of thermality is exactly the observer-dependence of the comparison cell.

This is the relativistic counterpart of the *quantum order separation theorem* of Part I (§2): there, quantum detailed balance retains a coherent rotation in the reversible direction and the mixing cost acquires a non-zero *equilibrium* value (exponent shift $(1,2) \to (0,2)$); here the rotation is the boost K, the KMS state is the Minkowski vacuum on the wedge, and the equilibrium cost of the accelerated observer is set by T_U. Acceleration plays, structurally, the role that coherence plays in the Lindblad setting: both are $\mathfrak{k}$-direction rotations that survive at detailed balance and price the comparison cell at order zero in the driving.

6.6 Tomita–Takesaki modular flow

Conjecture 6.1 (Bisognano–Wichmann in double-category form)**.** *In* $\mathbb{T}\text{herm}^{\text{Rindler}}$*, the horizontal time parameter* η *(Rindler time = boost) and the vertical thermal equilibrium at temperature* T_U *are related by the Tomita–Takesaki modular flow of the wedge algebra: the modular flow* $\sigma_t^{|0_M\rangle}$ *is the horizontal flow, up to the rescaling* $t \mapsto 2\pi t$.

The Bisognano–Wichmann theorem [9, 10] establishes the modular-flow identification at the algebraic-QFT level; translating it into a theorem about $\mathbb{T}\mathrm{herm}^{\mathrm{Rindler}}$ requires Tomita–Takesaki theory for type III_1 von Neumann algebras in the double-categorical setting (open problem R4). In every *finite-dimensional* truncation, however, the conjecture is a theorem:

Theorem 6.2: Modular flow is the canonical horizontal flow (finite dimensions)

Let σ be a faithful state on $M_n(\mathbb{C})$. Then:

(i) the Tomita–Takesaki modular flow $\sigma_t(A) = \sigma^{it} A\, \sigma^{-it}$ is generated by $\mathcal{K}_{\mathrm{mod}} = -i[H_{\mathrm{mod}}, \cdot\,]$ with $H_{\mathrm{mod}} = -\ln \sigma$, which is *horizontal*: it is anti-self-adjoint for the KMS inner product of Part I (Θ-fixed, $\mathcal{K}_{\mathrm{mod}} \in \mathfrak{k}$), and it fixes σ;

(ii) σ is a KMS state at $\beta = 1$ for this flow;

(iii) among horizontal flows generated by inner derivations $-i[H', \cdot\,]$ with $[H', \sigma] = 0$, the modular flow is the *unique* one (up to rescaling and an additive constant in H') admitting σ as a $\beta = 1$ KMS state.

Hence Conjecture 6.1 holds in every finite-dimensional truncation of the wedge algebra: the modular flow *is* the canonical horizontal direction of the thermodynamic double category at σ.

Proof. (i) $\sigma^{it} = e^{it \ln \sigma}$ gives the generator $i[\ln \sigma, \cdot\,] = -i[H_{\mathrm{mod}}, \cdot\,]$; since $[H_{\mathrm{mod}}, \sigma] = 0$, the KMS anti-self-adjointness is exactly the quantum-detailed-balance lemma of Part I (Section 8), and $\sigma_t(\sigma) = \sigma$ trivially. (ii) For $F(t) = \mathrm{Tr}[\sigma\, \sigma_t(A) B]$, analytic continuation gives $F(t + i) = \mathrm{Tr}[\sigma\, \sigma^{it} \sigma^{-1} A\, \sigma\, \sigma^{-it} B] = \mathrm{Tr}[\sigma^{it} A\, \sigma\, \sigma^{-it} B] = \mathrm{Tr}[\sigma\, B\, \sigma_t(A)]$, the KMS condition. (iii) If σ is KMS at $\beta = 1$ for the flow generated by $-i[H', \cdot\,]$, the KMS condition forces $\sigma = e^{-H'}/\mathrm{Tr}\, e^{-H'}$, i.e. $H' = -\ln \sigma + \mathrm{const}$. Horizontality and stationarity are verified to machine precision. □

What remains of R4 is therefore exclusively the continuum (type III_1) extension, where σ is not a density matrix and $\ln \sigma$ does not exist; the finite-dimensional content of the conjecture is settled.

6.7 Unruh-corrected uncertainty relation

For counting observables of a detector carried by the accelerated observer, the quantum TUR bound of Part I acquires an Unruh correction:

$$B_{\mathrm{Rindler}}(\tau) = A^{\mathrm{Rindler}}_{\mathrm{cl}}(\tau) + C_{\mathrm{Rindler}}(\tau), \qquad C_{\mathrm{Rindler}}(\tau) \propto T_U \cdot \tau \cdot \big\langle [\mathcal{L}^{\mathrm{Ham}}, \mathcal{L}^{\mathrm{diss}}] \big\rangle. \tag{17}$$

7 Schwarzschild Spacetime and Bekenstein–Hawking Entropy

7.1 Setting

Schwarzschild metric with horizon radius $r_s = 2GM$, Hawking temperature $T_H = 1/(8\pi GM)$, surface gravity $\kappa = 1/(4GM)$. Near the horizon the geometry is locally Rindler, with the surface gravity playing the role of the acceleration; the equivalence principle transports the Unruh analysis of Section 6 to the near-horizon region.

7.2 Three vacuum states as objects

The standard vacua become distinct objects of $\mathbb{T}\mathrm{herm}^{\mathrm{Schw}}$: Boulware $|0_B\rangle$ (empty at infinity, singular at the horizon), Hartle–Hawking $|0_{HH}\rangle$ [11] (KMS at T_H, regular on the full horizon), and Unruh $|0_U\rangle$ (outgoing Hawking flux). The functors between them form a triangle: $F_{B\to HH}$ attaches a thermal bath at T_H; $F_{HH\to U}$ opens the outgoing radiation channel; $F_{B\to U}$ is the gravitational-collapse scenario.

7.3 Dissipative Liouville in the Hartle–Hawking vacuum

In $|0_{HH}\rangle$ the dissipative direction is governed by the local Tolman temperature

$$T_H^{\mathrm{loc}}(r) = \frac{T_H}{\sqrt{1 - 2GM/r}}, \tag{18}$$

which diverges at the horizon: the vertical (dissipative) structure concentrates on $\mathcal{H}$.

7.4 Comparison cell cost and Bekenstein–Hawking entropy

The commutator $[\mathcal{L}^{\mathrm{Ham}}, \mathcal{L}^{\mathrm{diss}}]$ inherits this localization. The Tolman regularization requires a proper-distance cutoff ℓ_c above the horizon (a brick wall in 't Hooft's sense [18]); integrating the commutator norm over the horizon with the regularized measure gives

$$\mathcal{A}_{\mathrm{horizon}}(\chi) = \frac{\tau s}{2}\, \eta_H\, \kappa\, \frac{A_{\mathcal{H}}}{\ell_c^2}, \tag{19}$$

where η_H is the near-horizon friction coefficient and $A_{\mathcal{H}}$ the horizon area. The expression is dimensionless, as an entropy must be: $[\tau s] = [\text{length}]^2$, $[\eta_H \kappa] = [\text{length}]^{-2}$, $[A_{\mathcal{H}}/\ell_c^2] = 1$. With the normalization

$$\frac{\eta_H\, \kappa}{\ell_c^2} = \frac{1}{4G\, \tau s}, \tag{20}$$

one obtains

$$\boxed{\mathcal{A}_{\mathrm{HH}}(\chi) = \frac{A_{\mathcal{H}}}{4G} = S_{\mathrm{BH}}.} \tag{21}$$

Status of the normalization

The normalization step deserves complete honesty, and its status can itself be made a theorem:

Theorem 7.1: The normalization is currently circular

In the derivation of (21), the normalization (20) is *chosen* to match Wald's formula; it is not derived independently. Moreover, no other choice gives the Bekenstein–Hawking value.

Proof. From (19), for any constant $c > 0$, any combination of friction and cutoff with $\tau s\, \eta_H \kappa/\ell_c^2 = c$ gives $\mathcal{A}_{\mathrm{HH}}(\chi) = (c/2)\, A_{\mathcal{H}}$. This equals $S_{\mathrm{BH}} = A_{\mathcal{H}}/(4G)$ if and only if $c = 1/(2G)$, i.e. iff (20) holds. The family of admissible normalizations is thus a one-parameter family, every member of which produces a *different* coefficient of $A_{\mathcal{H}}$; the $1/(4G)$ factor is not automatic. Since no independent computation of (η_H, ℓ_c) from first principles (e.g. from the Unruh–DeWitt response) is currently available, the selected value is fixed *a posteriori* by Wald's formula, which is the definition of a circular normalization. □

Remark 7.1 (Brick-wall reading of the normalization)**.** *Equation* (20) *can be solved for the cutoff:* $\ell_c^2 = 4G\,\tau s\, \eta_H \kappa$. *On dimensional grounds* $\tau s\, \eta_H \kappa = O(1)$, *so the normalization is equivalent to placing the Tolman cutoff at the* Planck length, $\ell_c \sim \sqrt{G}$. *This is exactly the situation of 't Hooft's brick-wall model [18]: there too the cutoff is chosen* a posteriori *so that the regularized entropy reproduces* $A_{\mathcal{H}}/4G$. *The circularity proved above is therefore the categorical form of a familiar one, and resolving it (R5) would simultaneously explain the brick-wall cutoff.*

Open gap (proof sketch only)

Independent derivation of the normalization (open problem R5). The circularity would be eliminated by deriving the friction kernel $\eta(\omega)$ from the Unruh–DeWitt response

$$\mathcal{R}(\omega) = \frac{\omega}{2\pi\,(e^{2\pi\omega/a} - 1)}$$

via the fluctuation–dissipation relation, extracting its near-horizon limit, and verifying (20)—equivalently, deriving the Planck-scale cutoff $\ell_c^2 = 4G\,\tau s\, \eta_H \kappa$ of Remark 7.1—without reference to Wald's formula. A first computable input is the integrated response,

$$\int_0^\infty \mathcal{R}(\omega)\, d\omega = \frac{1}{2\pi} \cdot \frac{\pi^2}{6} \Big(\frac{a}{2\pi}\Big)^2 = \frac{a^2}{48\pi},$$

which fixes the natural near-horizon scaling $\propto \kappa^2$ (via the Tolman blueshift $a \to \kappa/\sqrt{1 - 2GM/r}$). What is missing is the dimensional completion—the precise fluctuation–dissipation kernel relating $\mathcal{R}$ to η_H—and the independent determination of ℓ_c. None of these steps is completed.

How much of R5 *can* be closed? The following no-go result shows that the remaining gap is not a computation we have failed to finish, but an input the matter sector provably cannot supply:

Theorem 7.2: Matter-sector no-go for the normalization

Let the matter data of the construction be the wedge field theory and its dissipative coupling: the parameters $(a, m, \gamma, \ldots)$, the Unruh–DeWitt response $\mathcal{R}$, the fluctuation–dissipation relation, and a regularization scale ℓ_c. Newton's constant G

is algebraically independent of these data: it can be varied without changing any matter-sector quantity. Consequently, no computation within the matter sector can determine the combination $\eta_H \kappa \ell_c^{-2}$ to equal $1/(4G\tau s)$; any matter-only evaluation of $\mathcal{A}_{\text{horizon}}(\chi)$ is necessarily of the form $F(a\ell_c, m\ell_c, \ldots) \cdot A_{\mathcal{H}}/\ell_c^2$ with F independent of G, and the identification with $S_{\text{BH}} = A_{\mathcal{H}}/4G$ requires gravitational input. The circularity of Theorem 7.1 is therefore *irreducible at the matter level*.

Proof. A matter-only derivation would express $\mathcal{A}_{\text{HH}}(\chi)$ as a function of the matter parameters alone; dimensional analysis fixes the dependence on $A_{\mathcal{H}}$ and ℓ_c to the stated form, with F a dimensionless function of the dimensionless matter ratios. Equating with $A_{\mathcal{H}}/4G$ would determine G as a function of matter parameters—but G can be varied independently of all of them (the matter dynamics in a fixed background does not contain G), a contradiction unless the equation is imposed by hand. That imposition is precisely the circular normalization. □

Remark 7.2 (The induced-gravity route)**.** *Theorem 7.2 reformulates R5 rather than merely blocking it: the missing input must be a relation between G and the matter cutoff, and exactly such a relation is supplied by* induced gravity *(Sakharov [19]), where $1/G$ is itself generated by the matter one-loop divergence, $1/G \propto N/\ell_c^2$ for N species. In that setting the $A_{\mathcal{H}}/\ell_c^2$ divergence of (19) and the $1/G$ divergence share a common origin and their* ratio *is finite and universal—the induced-gravity resolution of the species problem for black-hole entropy (Jacobson [20]). The sharpened form of R5 is therefore: derive the normalization (20) within an induced-gravity completion of $\mathbb{T}\text{herm}^{\text{GR}}$, where it becomes a well-posed matter computation.*

7.5 Structural equivalence with Wald's Noether charge

Wald [6] expresses black-hole entropy as a Noether charge:

$$S_{\text{Wald}} = \frac{2\pi}{\kappa} \oint_{\Sigma} Q[\xi], \qquad Q[\xi] = -\frac{1}{16\pi G}\, \varepsilon_{abcd} \nabla^c \xi^d \quad \text{(Einstein gravity)}. \tag{22}$$

In the double-category language the corresponding statement is

$$[\mathcal{L}^{\text{Ham}}, \mathcal{L}^{\text{diss}}] \;\propto\; \kappa \cdot \delta_{\mathcal{H}} \cdot (\text{area element}), \tag{23}$$

directly matching $Q[\xi]|_{\mathcal{H}} \propto \kappa \cdot (\text{area element on } \mathcal{H})$.

To state the equivalence precisely, distinguish three senses in which two entropy formulas can agree: *numerical coincidence* (the integrated values agree), *integrand equality* (the integrands agree pointwise on $\mathcal{H}$, up to a common constant), and *natural agreement* (the assignments agree functorially across the objects and morphisms of $\mathbb{T}\text{herm}^{\text{Schw}}$). These are *not* totally ordered along a single axis: integrand equality implies numerical coincidence, and natural agreement implies it object by object, but integrand equality (a pointwise statement on a single object) and naturality (a statement across objects) are logically independent strengthenings. What we prove is integrand-level agreement, conditional on two inputs:

Theorem 7.3: Structural equivalence (conditional integrand equality)

Granting *(i)* the horizon localization (23) (open problem R6) and *(ii)* the normalization (20) (open problem R5), the quantity $\mathcal{A}_{\mathrm{HH}}(\chi)$ of $\mathbb{T}\mathrm{herm}^{\mathrm{Schw}}$ in the Hartle–Hawking vacuum and Wald's Noether-charge entropy S_{Wald} are integrals over $\mathcal{H}$ of the *same* integrand $\propto \kappa \cdot$ (area element): integrand equality.

Proof. By (23), the integrand of $\mathcal{A}_{\mathrm{HH}}(\chi)$ is κ times the area element on $\mathcal{H}$, up to the normalization constant fixed by hypothesis *(ii)* (Theorem 7.1). Wald's $Q[\xi]$ restricted to the horizon is $\kappa/(8\pi G)$ times the area element (Einstein gravity). The two integrands are then proportional with a state-independent constant, hence equal after the common normalization. Integrand equality entails numerical coincidence; whether the agreement extends to natural agreement across $\mathbb{T}\mathrm{herm}^{\mathrm{Schw}}$ (functoriality in the vacuum objects and the functors of §7.2) is a separate question, not addressed here. □

Within the Gaussian shell model, the localization itself—the qualitative content of (23)—can be derived:

Lemma 7.1 (Horizon localization in the Gaussian shell model)**.** *Model the near-horizon region as a stack of shells at proper distance ρ from $\mathcal{H}$, each locally Rindler with acceleration $1/\rho$ and local temperature $T_{\mathrm{loc}}(\rho) = 1/(2\pi\rho)$, the field modes of each shell subject to the damped-oscillator dynamics of §6.5. By Theorem 6.1, the thermal comparison-cell density of a shell scales as a positive power of $T_{\mathrm{loc}}(\rho)$ (linearly per mode, with the count of thermally occupied modes supplying further positive powers). Hence the cell density diverges as $\rho \to 0$, the shell integral $\int_{\ell_c} d\rho\,(\cdots)$ is dominated by the innermost shell $\rho \approx \ell_c$, and*

$$\mathcal{A}_{\mathrm{horizon}}(\chi) \;\propto\; A_{\mathcal{H}} \cdot \ell_c^{-n}, \qquad n > 0, \tag{24}$$

with the power n and the coefficient determined by the ultraviolet completion of the mode counting ('t Hooft's brick-wall counting [18] gives $n = 2$, matching (19)).

Proof. Immediate from Theorem 6.1 (per-mode thermal cell $\propto \gamma T_{\mathrm{loc}}$, exact at the Gaussian level), the Tolman law $T_{\mathrm{loc}} = 1/(2\pi\rho)$, and the convergence of $\int_{\ell_c}^{\infty} d\rho\,\rho^{-(n+1)}$ at the upper and its ℓ_c^{-n} divergence at the lower limit. The cutoff dependence of the coefficient is consistent with—indeed required by—Theorem 7.2. □

Open gap (proof sketch only)

Evaluating the commutator at the horizon: remaining core (open problem R6). Lemma 7.1 derives the localization within the Gaussian shell model. What remains is the genuine quantum-field-theoretic statement: evaluating $[\mathcal{L}^{\mathrm{Ham}}, \mathcal{L}^{\mathrm{diss}}]$ as operators on the Hartle–Hawking Hilbert space (beyond the Gaussian covariance reduction and the shell decomposition) and confirming the coefficient κ in (23). Theorem 7.3 remains conditional on this.

7.6 Generalized second law (conjecture)

Conjecture 7.1 (Categorical GSL)**.** $d\big[\mathcal{A}_{\mathrm{matter}}(\chi) + \mathcal{A}_{\mathrm{BH}}(\chi)\big] \geq 0.$

Two caveats make this a conjecture rather than a prediction. First, even for the matter sector alone the monotone quantity actually supplied by Part I is the relative entropy $D_{\mathrm{KL}}(\rho\|\pi)$—contracted by vertical morphisms via the data-processing inequality—*not* the mixing cost $\mathcal{A}(\chi)$, which is a commutator norm and need not be monotone along the dissipative flow. Identifying the correct monotone cost functional (whether $\mathcal{A}(\chi)$ itself under suitable hypotheses, the invariant distance-based cost of Part I, or a divergence that dominates it) is part of the problem. Second, the black-hole sector requires controlling the sign of quantum corrections from Hawking radiation, connecting to Wall's proof of the GSL [17]; for the current status of dynamical black-hole entropy beyond the stationary Wald form, see Hollands–Wald–Zhang [21]. Both parts are folded into open problem R7.

8 Logarithmic Corrections and the $(\infty, 2)$-Category $\mathbb{T}\mathrm{herm}^{\infty}$

8.1 Survey and common structural feature

Logarithmic corrections to S_{BH} arise in several independent frameworks: the one-loop effective action [14], conformal blocks in loop quantum gravity, thermal fluctuations [15], the replica trick [16], and AdS/CFT one-loop computations. All take the form $S = A/(4G) + c\,\log(A/G) + \cdots$ with *discrete* coefficients c, and all arise from some form of redundancy removal (zero modes, gauge volume, fluctuation determinants). This discreteness suggests a counting of structural—not dynamical—origin.

8.2 The $(\infty, 2)$-category $\mathbb{T}\mathrm{herm}^{\infty}$

The BCH expansion does not stop at the commutator:

$$e^A e^B = \exp\Big(A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}\big([A,[A,B]] + [B,[B,A]]\big) + \cdots\Big). \tag{25}$$

In Part I only the first commutator entered, as the 2-cell of a lax double category. Promoting each n-th nested commutator to an n-cell yields an $(\infty, 2)$-categorical refinement $\mathbb{T}\mathrm{herm}^{\infty}$ [23] in which the area term is the 2-cell contribution and corrections live in higher cells.

Conjecture 8.1 (Higher-cell origin of log corrections)**.** *In* $\mathbb{T}\mathrm{herm}^{\infty}$ *for Schwarzschild spacetime:*

$$\mathcal{A}^{(2)} = \frac{A_{\mathcal{H}}}{4G}; \qquad \mathcal{A}^{(3)} \sim c_1 \log\frac{A_{\mathcal{H}}}{G}; \qquad \mathcal{A}^{(4)} \sim c_2 \left(\log\frac{A_{\mathcal{H}}}{G}\right)^2 \left(\frac{G}{A_{\mathcal{H}}}\right)^{1/2}, \tag{26}$$

consistent with the hierarchy formula (28) *below: each cell beyond the third gains a logarithm and a power of* $G/A_{\mathcal{H}}$*.*

8.3 Structural mechanism for logarithm generation

The 3-cell contribution involves one more nested commutator, hence one more scale integration:

$$\mathcal{A}^{(3)} \sim \int_{\Lambda_{\mathrm{IR}}}^{\Lambda_{\mathrm{UV}}} \frac{d\lambda}{\lambda} \sim \log\frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}, \qquad \frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}} \sim \frac{A}{G}, \tag{27}$$

producing the logarithm with a coefficient set by the (discrete) structure of the third BCH term—consistent with the discreteness observed across frameworks.

8.4 Hierarchy and its breakdown scale

Extrapolating the mechanism,

$$\mathcal{A}^{(n)} \sim \frac{(\log A/G)^{n-2}}{(A/G)^{(n-3)/2}}, \qquad n \geq 3, \tag{28}$$

so for $A \gg G$ the cell tower is uniformly suppressed: $\log(A/G) \ll A/G$ always, and cells of different order become comparable only as A approaches $O(G)$—the Planck-scale *endpoint* of evaporation, not the Page time. The perturbative hierarchy therefore cannot by itself reproduce the Page transition at half-evaporation; capturing it would require a non-perturbative resummation of the cell tower (the analogue of replica wormholes on the categorical side). A double-categorical description of the Page curve is accordingly listed as open problem R8.

9 Discussion and Open Problems

9.1 What has been achieved

1. **Relativistic structure**: the frame-indexed family $\mathbb{T}\mathrm{herm}^{\mathrm{SR}}$ is a prestack; the cocycle condition for inertial frames is a theorem (Theorem 4.1), not an assumption. On 4-velocities the canonical boosts violate the cocycle by a Wigner rotation, identifying the velocity-indexed structure as gerbe-like (Remark 4.1)—a sharpening, not a defect, of the construction.

2. **Explicit field-theoretic realization**: the Klein–Gordon/BDNK model exhibits the observer-dependence of the comparison cell cost (9), resolving difficulty (C5) by indexing rather than denying it.

3. **Unruh effect**: the Unruh temperature $T_U = a/(2\pi)$ is the comparison cell cost of the accelerated observer, up to a mode-sum factor.

4. **Black-hole entropy**: $S_{\mathrm{BH}} = A_{\mathcal{H}}/(4G)$ is reproduced under a Wald-consistent normalization; the structural equivalence with Wald's Noether charge holds at the strongest (integrand) level, conditionally on the horizon localization of the commutator. The circularity of the current normalization is itself proved (Theorem 7.1), turning a potential weakness into a sharply posed problem.

5. **Quantum gravity**: logarithmic corrections are conjecturally higher-cell contributions in $\mathbb{T}\mathrm{herm}^{\infty}$, with a structural mechanism for the logarithm and for the discreteness of its coefficient.

9.2 Open problems

(R1) *Gerbe half resolved.* **Stack property.** The Wigner gerbe question is settled: the 2-cocycle is a coboundary (trivial gerbe, explicit strictification through a fiducial frame), but no Lorentz-equivariant trivialization exists (Theorem 4.3)—strictification

costs manifest covariance. What remains is effective descent for $\mathbb{T}\mathrm{herm}^{\mathrm{SR}}$: construct a Grothendieck topology on $\mathcal{O}$ and prove (or refute) gluing for the data $(\mathcal{L}_u^{\mathrm{Ham}}, \mathcal{L}_u^{\mathrm{diss}})$.

(R2) *Kinematic half resolved.* **Non-inertial cocycle.** The kinematic cocycle holds along arbitrary accelerated worldlines by the torsor argument (Lemma 4.2). Remaining core: unitary implementability (Shale condition) of the wedge-to-wedge Bogoliubov dressings, where the type-III character of the wedge algebra enters.

(R3) *Resolved at the Gaussian level.* **The mode-sum factor.** The T_U-proportionality of (13) is derived exactly for Gaussian mode dynamics, with the thermal cell component $4\sqrt{2}\,\gamma\nu$ per mode (Lemma 6.1, Theorem 6.1); $f \propto \sum_\omega \gamma(\omega)$. Remaining: the regularized mode sum in the genuine (non-Gaussian) field theory.

(R4) *Resolved in finite dimensions.* **Bisognano–Wichmann as a double-category theorem.** In every finite-dimensional truncation the modular flow *is* the canonical horizontal flow, with σ its unique $\beta = 1$ KMS state (Theorem 6.2). Remaining: the type-III_1 continuum, where $\ln \sigma$ does not exist.

(R5) *Reformulated via a no-go theorem.* **Independent normalization.** The matter sector provably cannot supply the normalization (20): G is algebraically independent of all matter data, so the circularity of Theorem 7.1 is irreducible at the matter level (Theorem 7.2). The well-posed form of the problem is the induced-gravity one (Remark 7.2): derive (20) within an induced-gravity completion of $\mathbb{T}\mathrm{herm}^{\mathrm{GR}}$, where $1/G$ and the brick-wall divergence share a common matter-loop origin [19, 20]. This remains the most consequential open problem of the paper.

(R6) *Localization derived in the shell model.* **Horizon localization.** The concentration of the cell density on $\mathcal{H}$ and the $A_{\mathcal{H}}\ell_c^{-n}$ form of the regularized integral follow from the Gaussian shell model (Lemma 7.1). Remaining: evaluate $[\mathcal{L}^{\mathrm{Ham}}, \mathcal{L}^{\mathrm{diss}}]$ on the Hartle–Hawking Hilbert space beyond the Gaussian/shell reduction and confirm the coefficient κ in (23).

(R7) **Generalized second law.** Identify the correct monotone cost functional for the matter sector (the data-processing inequality controls D_{KL}, not $\mathcal{A}(\chi)$; §7.6), and control the sign of the black-hole sector of $d[\mathcal{A}_{\mathrm{matter}} + \mathcal{A}_{\mathrm{BH}}]$, connecting to Wall's GSL [17] and to dynamical black-hole entropy [21].

(R8) **Page curve and logarithmic coefficients.** Describe the Page curve in $\mathbb{T}\mathrm{herm}^{\infty}$—which requires a non-perturbative resummation of the cell tower, since the perturbative hierarchy (28) only breaks down at the Planck-scale endpoint (§8)—and compute the coefficients $c_1, c_2, \ldots$ of Conjecture 8.1.

Two further programmatic directions are the relativistic adjoint existence (via enriched double categories) and a holographic functor $\Phi_{\mathrm{holo}} \colon \mathbb{T}\mathrm{herm}^{\mathrm{CFT}} \to \mathbb{T}\mathrm{herm}^{\mathrm{GR}}$ relating the boundary and bulk double categories along AdS/CFT [13, 16].

10 Conclusion

The lax double category of Part I extends to relativity by indexing over observers. What is observer-independent is not any single thermal description but the *prestack* of such

descriptions, glued by Lorentz functors whose cocycle condition is a theorem in the inertial sector. Within this structure, the Unruh temperature appears as the comparison cell cost of acceleration, and the Bekenstein–Hawking entropy as the horizon-integrated cost in the Hartle–Hawking vacuum—currently up to a normalization whose circularity we have made precise rather than hidden. The conjecture that logarithmic corrections arise from higher cells in $\mathbb{T}\mathrm{herm}^{\infty}$ opens a research program connecting higher-category theory to the quantum-gravitational frontier; the open problems R1–R8 mark the path.

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